NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. EM-74

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CONTRACT No. AF-19(122)-42

FEBRUARY 1955

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The research reported in this document has been made possible through support and sponsorship extended by the Air Force Cambridge Research Center, under Contract No. AF-19(122)-42. It is published for technical information only and does not necessarily represent recommendations or conclusions of the sponsoring agency.

New York, 1955

Abstract

The differential equation $u''(k,x) + [k^2-V(x)]\,u(k,x) = 0$ is considered in the interval $-\infty < x < \infty$. From a knowledge of the reflection coefficient $b(k)$ associated with a wave e^{ikx} incident from the left it is shown how one can calculate the function $V(x)$ of the differential equation (when $V(x) = 0$ for x less than some constant). The method used is due to I.M. Gelfand and B.M. Levitan.

Sufficient conditions such that a function $b(k)$ can be the reflection coefficient corresponding to a differential equation of this form are given.

An example, in which $b(k) = -(k^2+1)^{-1}$, is worked out in detail. It is also shown how one can obtain $V(x)$ explicitly for a large class of reflection coefficients.

The differential equation $u''(k,x) + k^2n^2(x)\,u(k,x) = 0$ in the interval $-\infty < x < \infty$ is also considered, the problem being to determine the function $n(x)$ from a knowledge of the reflection coefficient, assuming that $n(x) =$ constant for x less than some given value.

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1. Introduction

It is a customary procedure in mathematical physics to make predictions about particular physical situations on the basis of a general theory and then to compare those predictions with experimental data. This procedure is, of course, necessary to check the validity of a new theory. But when a physical theory has evoked sufficient confidence because of its continuing success over many years one need no longer look upon such a procedure as a test of the theory. Rather, the theory becomes a tool by means of which one can describe in detail the various special cases arising when particular conditions are added to the general theoretical assumptions. Experimental data for a special case are then useful to measure or make determinate those physical quantities of the general theory which otherwise are arbitrary. In the application of a trusted theory to special cases it still seems to be customary, however, to proceed in the same way as in the verification of a new theory. One makes certain assumptions about the relevant physical situation, one predicts the consequences of these assumptions on the basis of the general theory, and finally one makes an empirical check of the assumptions by means of experimental data. The assumptions can be corrected according to the experimental results and the procedure repeated until the desired accuracy is reached. In the case of an old and tested theory, however, it would appear more logical to adopt a different procedure, if possible. A gain in time, accuracy and understanding might be expected if one were able to reconstruct directly from the experimental data the physical conditions corresponding to a special case of the general theory.

The problem of making such a reconstruction is the direct inverse of the problem ordinarily undertaken. This reconstruction is closely related to the problem of synthesis wherein the task is to construct special physical con-

ditions which, according to the general theory, will produce experimental data given beforehand.

An example of the 'inverse problem' is the attempt to determine the ionization density of the ionosphere from the time delay of a pulse radio wave has been transmitted from the earth and relected back to the earth by the ionosphere. The synthesis problem is a common one for electical engineers who are often required to design networks which produce a prescribed response to a prescribed input voltage over a given frequency band.

Some work has been done in recent years on inverse problems suggested by quantum mechanics and associated with the differential equation

$$\frac{d^2}{dx^2} u(k,x) + \left[k^2 - V(x)\right] u(k,x) = 0 .$$

For the quantum mechanical application the boundary condition $u(k,0) = 0$ is assumed and the variable x lies in the interval $0 \leq x < \infty$. The given experimental data provide a knowledge of the phase function $\phi(k)$ which appears in the asymptotic form of $u(k,x)$ for large positive x:

$$u(k,x) \sim A(k) \sin\left[kx + \phi(k)\right], \qquad x \to +\infty .$$

The problem is: given $\phi(k)$; determine $V(x)$. The uniqueness of the solution when the operator $-\frac{d^2}{dx^2} + V(x)$ has no bound states (proper eigenfunctions) was proved by N. Levinson[1] and V.A. Marcenko[2]. A similar inverse problem for the same differential equation arises when a finite interval is chosen and homogeneous boundary conditions are applied at both ends: given the two sets of eigenvalues of the operator $-\frac{d^2}{dx^2} + V(x)$ corresponding to two different pairs of boundary conditions; determine $V(x)$. The uniqueness of $V(x)$ in this situation was proved by N. Levinson[3] and G. Borg[4] and V.A. Marcenko[2]. I.M. Gelfand and B.M. Levitan[5] have shown how to construct the quantity $V(x)$

for the interval $0 \leq x < \infty$ from the spectral measure function associated with the operator $-\frac{d^2}{dx^2} + V(x)$ and boundary conditions. Another solution by an entirely different method, capable of including the case in which bound states exist, was given by W. Kohn and R. Yost [6]. M. Krein [7] has solved the inverse problem in a different way for the finite interval.

A very clear English summary of the paper by I.M. Gelfand and B.M. Levitan [5], with an additional note on the error due to a small perturbation in the spectral measure function, has been published by N. Levinson [8].

It will be the purpose of this paper to treat the same ordinary differential equation in the interval $-\infty < x < \infty$. The information assumed given will be the reflection coefficient $b(k)$ when the solution $u(k,x)$ of the differential equation has the asymptotic forms

$$u(k,x) \to e^{ikx} + b(k)\, e^{-ikr}, \qquad x \to -\infty,$$

$$u(k,x) \to a(k)\, e^{ikx}, \qquad x \to +\infty .$$

The function $V(x)$ will be obtained from $b(k)$ by an adaptation of the method of I.M. Gelfand and B.M. Levitan [5], and an explicit example will be worked out. Moreover, sufficient conditions will be derived for a function $b(k)$ to be the reflection coefficient of such a differential equation (this is necessary for the synthesis problem), and a method for solving a large class of inverse problems explicitly will be given. Finally, the differential equation

$$\frac{d^2}{dx^2} u(k,x) + k^2\, n^2(x)\, u(k,x) = 0,$$

for which the problem is to find the function $n(x)$, will be considered.

The solutions obtained in this article appear to have number of applications. The synthesis problems for a non-uniform transmission line (in which either the inductance or the capacitance is constant) and, under certain con-

ditions, for a waveguide of variable cross-section can be treated by the method described here. A synthesis problem in which a variable stratified dielectric is to be constructed so that the reflection coefficient of an electromagnetic wave will vary in a prescribed way with frequency can also be treated. Another synthesis problem involves a plane wave incident on a plane interface of a variable dielectric slab; the reflected wave amplitude variation as a function of the angle of incidence is prescribed, and the variation of the dielectric properties of the slab can be found so as to produce the prescribed reflected wave amplitude variation. The problem, already mentioned, of determining the ionization density of a stratified ionosphere is another possible application.

In this investigation the preliminary information used concerning the differential equation

$$\frac{d^2}{dx^2} u(k,x) + \left[k^2 - V(x)\, u(k,x)\right] = 0$$

and the general facts about the spectral theory and about the scattering matrix associated with the differential equation in the interval $-\infty < x < \infty$ can be found in the Appendix 17 of K. Friedrich's notes on the spectral theory of linear operators in Hilbert space[9].

2. The basic differential equation and some of its properties

If a continuously differentiable wave function $u(k,x)$ satisfies the differential equation

$$\frac{d^2u(k,x)}{dx^2} + \left[k^2 - V(x)\right] u(k,x) = 0, \tag{2.1}$$

where the function $V(x)$ is real, bounded, piecewise continuous and belongs to L_1 in the interval $-\infty < x < \infty$, then it is well known that the asymptotic behavior of $u(k,x)$ can be expressed in the form

$$\begin{aligned} u(k,x) &\sim \gamma_+(k)\, e^{ikx} + \delta_+(k)\, e^{-ikx}, \qquad x \to +\infty \\ u(k,x) &\sim \gamma_-(k)\, e^{ikx} + \delta_-(k)\, e^{-ikx}, \qquad x \to -\infty . \end{aligned} \tag{2.2}$$

The quantities $\gamma_{\pm}$, $\delta_{\pm}$ are, as indicated, functions of k alone and independent of x. Moreover, a linear relation exists between the components of the vector $\begin{pmatrix}\gamma_+\\ \delta_+\end{pmatrix}$ and those of the vector $\begin{pmatrix}\gamma_-\\ \delta_-\end{pmatrix}$. This linear relation can be expressed in terms of a two by two square matrix, the so-called scattering matrix S: $\begin{pmatrix}\gamma_-\\ \delta_-\end{pmatrix} = S\begin{pmatrix}\gamma_+\\ \delta_+\end{pmatrix}$. The four components of S are functions of k alone which are the same for all solutions $u(k,x)$ of (2.1) and are fixed by the function $V(x)$ appearing in (2.1). If we write S in the form

$$S = \begin{pmatrix}\alpha(k) & \alpha'(k)\\ \beta(k) & \beta'(k)\end{pmatrix}, \tag{2.3}$$

the following relations between the components of S hold for real k:

$$\begin{aligned} \alpha(-k) &= \overline{\alpha(k)} = \alpha'(k) \\ \beta(-k) &= \overline{\beta(k)} = \beta'(k) \\ |\alpha|^2 &- |\beta|^2 = 1 . \end{aligned} \tag{2.4}$$

From the last equation in (2.4) it follows that

$$(2.5) \qquad |\alpha|^2 > |\beta|^2; \quad |\alpha|^2 \geq 1 ,$$

for finite values of $|\alpha|$ and $|\beta|$ when k is real. It is also known that $\alpha(k)$ and $\beta(k)$ are analytic functions of k in the complex plane.

It is possible to find a solution of (2.1) which has the following asymptotic behavior:

$$(2.6) \qquad \begin{aligned} u(k,x) &= e^{ikx} + b(k)e^{-ikx}, && x \to -\infty \\ &= a(k)\, e^{ikx} , && x \to +\infty. \end{aligned}$$

From the definition of the scattering matrix we have

$$(2.7) \qquad b(k) = \frac{\beta(k)}{\alpha(k)}, \quad a(k) = \frac{1}{\alpha(k)} ,$$

and from (2.4) it follows that

$$(2.8) \qquad |a|^2 + |b|^2 = 1$$

when k is real. In accordance with the asymptotic behavior of u(k,x) given by (2.6) the quantity b(k) is called the reflection coefficient and the quantity a(k) the transmission coefficient.

In general, for the interval $-\infty < x < +\infty$, the operator $-\frac{d^2}{dx^2} + V(x)$ of equation (2.1) has a spectral representation with a spectrum composed of a discrete part and a continuous part. The discrete spectrum has for its corresponding eigenfunctions those solutions of (2.1) which belong to L_2. For values of k in the discrete spectrum the eigenfunctions must then die out exponentially at $+\infty$ and $-\infty$; therefore, from (2.2) it is apparent that the discrete eigenvalues occur only for values of k which have non-zero imaginary parts. Since the eigenvalues of (2.1) are given by k^2 which is real only when k is pure imaginary or real and since the operator $-\frac{d^2}{dx^2} + V(x)$ is Hermitian, it follows that the values

of k corresponding to the eigenvalues must lie on the imaginary axis. Since when $u(k,x)$ satisfies (2.1) so does $u(-k,x)$, we can without loss of generality restrict the eigenvalues to lie on the positive imaginary axis.

Further, the eigenvalues occur for values of k for which the function $b(k)$ has poles on the positive imaginary axis. It is known that these poles are simple and that $b(k)$ is otherwise regular in the upper half-plane and on the real axis, except perhaps at zero. If we add the condition that $xV(x)$ belong to L_1, then $b(k)$ has only a finite number of poles in the upper half-plane; i.e., there are only a finite number of eigenvalues.

The completeness of the spectral resolution of the operator $-\frac{d^2}{dx^2} + V(x)$ of (2.1) can be expressed succinctly by the relation

$$(2.9)\quad \sum_{\nu} u(k_\nu,x)\, u^*(k_\nu,y) + \int_{-\infty}^{\infty} u(k,x)\, u^*(k,y)\, dk = \delta(x-y) \ ,$$

where $\delta(x-y)$ is the Dirac delta-function which has the property that

$$\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = f(0)$$

for arbitrary $f(x)$. The function $u^*(k,x)$ is a particular solution of (2.1) corresponding to the given solution $u(k,x)$; the sum in (2.9) is taken over the values of k corresponding to the eigenvalues. When $u(k,x)$ and $u^*(k,x)$ are analytic functions of k the relation on the left side of (2.9) can be written as a single integral in the complex plane,

$$\int_C u(k,x)\, u^*(k,y)\, dk \quad ,$$

in which the contour of integration, C, is a large semi-circle in the upper half-plane. The relationship between the proper eigenfunctions and the improper

eigenfunctions [$u(k,x)$, k in the continuous spectrum] is made clear by this contour integral representation.

3. Formulation of the main problem and derivation of the fundamental integral equation

The problem we are going to consider is: given the reflection coefficient $b(k)$ as a function of k for an equation of the form (2.1) with an unknown real $V(x)$; to find $V(x)$ as a function of x. The method we shall use for this investigation is due to Gelfand and Levitan[5] and has been summarized by M. Levinson[8] in English.

The quantity $V(x)$ in (2.1) can be looked upon as a perturbation operator which, when added to the operator $-\frac{d^2}{dx^2}$ of the equation

$$\frac{d^2}{dx^2} u_o(k,x) + k^2 u_o(k,x) = 0 , \tag{3.1}$$

produces the operator $-\frac{d^2}{dx^2} + V(x)$ of (2.1). We shall refer to functions $u_o(k,x)$ satisfying (3.1) as solutions of the unperturbed equation, in contrast to the functions $u(k,x)$ which are solutions of the perturbed equation (2.1). From this point on, for simplicity, we shall assume that the function $V(x)$ vanishes for $x < -\frac{\delta}{2}$, where δ is some real positive constant.

The first step in the Gelfand-Levitan method is to recognize a fundamental property of the relationship between a solution $u_o(k,x)$ of the unperturbed equation and a solution $u(k,x)$ of the perturbed equation. There exists a linear transformation $T(x,y)$ which transforms a solution $u_o(k,x)$ of the unperturbed equation into a solution $u(k,x)$ of the perturbed equation, namely,

$$T(x,y) = \int_{-\infty}^{\infty} u(k,x)\, u_o^*(k,y)\, dk$$

where $u_o^*(k,y)$ is the conjugate function of $u_o(k,y)$ in the completeness relation

$$\int_{-\infty}^{\infty} u_o(k,x)\, u_o^*(k,y)\, dk = \delta(x-y).$$

The fundamental observation to be made is this: if $u(k,x) = u(k,x)$ for $x \leq -\delta$ $\left[\text{we recall that } V(x) = 0 \text{ for } x < -\frac{\delta}{2}\right]$, then $T(x,y)$ is <u>triangular</u> in the sense that $T(x,y) = 0$ for $y > x$. It is convenient to write $T(x,y) = \delta(x-y) + K(x,y)$. Then $K(x,y)$ is also triangular, that is, $K(x,y) = 0$ for $y > x$. We now state the following:

<u>Theorem</u> If $u(k,x) = u_o(k,x)$ for $x \leq -\frac{\delta}{2}$, then

$$u(k,x) = u_o(k,x) + \int_{-(x+\delta)}^{x} K(x,y)\, u_o(k,y)\, dy\,, \tag{3.2}$$

and $K(x,y)$ is uniformly bounded, continuous, and has piecewise continuous first and second derivatives.

To prove the theorem we apply the operator $\frac{d^2}{dx^2} - V(x)$ to (3.2) on both sides. Using (2.1) and (3.1) we obtain

$$\begin{aligned}
-k^2u(k,x) = {} & -k^2u_o(k,x) - V(x)u_o(k,x) + u_o(k,x)\,\frac{dK(x,x)}{dx} + K(x,x)\,\frac{du_o(k,x)}{dx} \\
& + u_o(k,-x-\delta)\,\frac{dK(x,-x-\delta)}{dx} + K(x,-x-\delta)\,\frac{du_o(k,-x-\delta)}{dx} + K_x(x,x)u_o(k,x) \\
& + K_x(x,-x-\delta)u_o(k,-x-\delta) + \int_{-(x+\delta)}^{x} K_{xx}(x,y)u_o(k,y)dy \\
& - V(x)\int_{-(x+\delta)}^{x} K(x,y)\, u_o(k,y)\, dy.
\end{aligned}$$

But from (3.2) and (3.1), integrating by parts, we have

$$-k^2u(k,u) = -k^2u_o(k,x) + \int_{-(x+\delta)}^{x} K(x,y)\,\frac{d^2}{dy^2}\,u_o(k,y)\,dy$$

$$= -k^2u_o(k,y) + K(x,x)\,\frac{du_o(k,x)}{dx} + K(x,-x-\delta)\,\frac{du_o(k,-x-\delta)}{dx}$$

$$-K_y(x,x)u_o(k,x) + K_y(x,-x-\delta)\,u_o(k,-x-\delta)$$

$$+ \int_{-(x+\delta)}^{x} K_{yy}(x,y)\,u_o(k,y)\,dy\;.$$

Then combining these two results and combining terms appropriately we obtain

$$0 = \int_{-(x+\delta)}^{x} \left\{K_{xx}(x,y)-K_{yy}(x,y)-V(x)K(x,y)\right\} u_o(k,y)\,dy$$

$$+ \left\{2\,\frac{dK(x,x)}{dx} - V(x)\right\} u_o(k,x) + \frac{dK(x,-x-\delta)}{dx}\quad u_o(k,x)\;.$$

This equation will be satisfied if

$$K_{xx}(x,y) - K_{yy}(x,y) - V(x)K(x,y) = 0$$

and

$$\frac{dK(x,x)}{dx} = \frac{1}{2}\,V(x), \tag{3.3}$$

and if,

$$\frac{dK(x,-x-\delta)}{dx} = 0,$$

that is, if $K(x,-x-\delta)$ = constant.

On the other hand we know from the theory of hyperbolic partial differential equations that there exists a $K(x,y)$ satisfying (3.3) (this is a characteristic boundary-value problem). It is easy to see that having determined $K(x,y)$ from (3.3) we could retrace our steps and arrive at (3.2). Thus

the existence of $K(x,y)$ is proved (obviously this function satisfies the conditions of the theorem). It should be observed that, according to (3.3), $K(x,y)$ is independent of the choice of particular solutions $u(k,x)$ of (2.1) and $u_0(k,x)$ of (3.1) except that (3.2) implies: $u(k,x) = u_0(kx)$ when $x \leq -\frac{\delta}{2}$. Thus $K(k,x)$ transforms any solution of (3.1) into a solution of (2.1) which is the same function for $x \leq -\frac{\delta}{2}$.

By a completely analogous argument we can demonstrate the existence of a triangular operator kernel $K_0(x,y)$ such that

$$u_0(k,x) = u(k,x) + \int_{-(x+\delta)}^{x} K_0(x,y)\, u(k,y)\, dy, \tag{3.4}$$

when $u(k,x) = u_0(k,x)$ for $x \leq -\frac{\delta}{2}$.

In order to make use of contour integration in the complex plane we shall work with two solutions $v(k,x)$ and $\hat{v}(k,x)$ of (2.1) which are analytic functions of k for $\mathrm{Im}\, K > 0$ and continuous functions of k for $\mathrm{Im}\, k \geq 0$. They can be defined by their asymptotic behavior:

$$v(k,x) \begin{cases} \sim \alpha(k)e^{ikx} + \beta(k)e^{-ikx}, & x \to -\infty \\ \sim e^{ikx}, & x \to +\infty, \end{cases}$$

(3.5)

$$\hat{v}(k,x) \begin{cases} \sim e^{-ikx} \\ \sim -\overline{\beta(k)}\ e^{ikx} + \alpha(k)\ e^{-ikx}, & x \to +\infty \end{cases} .$$

From the spectral representation theory of the operator $-\frac{d^2}{dx^2} + V(x)$ we have the relation

$$\frac{1}{2\pi} \int_C \frac{v(k,x)\hat{v}(k,y)}{\alpha(k)}\, dk = \delta(x-y)\ , \tag{3.6}$$

where the contour C is a large semi-circle in the upper half of the complex k-plane and the integration direction is clockwise. The completeness relation (3.6) could also be written

$$(3.7) \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{v(k,x)\hat{v}(k,y)}{\alpha(k)}\, dk - \sum_{\nu} a_\nu v(k_\nu,x)\hat{v}(k_\nu,y) = \delta(x-y) \, ,$$

where the sum runs over the poles of $\frac{1}{\alpha(k)}$ in the upper half-plane, and the coefficients a_ν are i times the residues $\frac{1}{\alpha'(k)}$ of $\frac{1}{\alpha(k)}$ at these poles.

Before we can proceed we must establish the following identity:

$$(3.8) \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{v(k,x)}{\alpha(k)}\, e^{-iky}\, dk - \sum_{\nu} a_\nu v(k_\nu,x)\, e^{-ik_\nu y} = 0, \qquad x > y \, .$$

To this end we consider the kernel $\hat{K}(x,y)$ in the relation

$$(3.9) \qquad e^{-iky} = \hat{v}(k,y) + \int_{-(y+\delta)}^{y} \hat{K}(y,z)\hat{v}(k,z)\, dz \, .$$

Assume $x > y$. Multiply (3.9) by $\frac{v(k,x)}{2\pi\, \alpha(k)}$ and integrate with respect to k from $-\infty$ to $+\infty$; then add to this the result of multiplying (3.9) with $k = k_\nu$ by $a_\nu u(k_\nu,x)$ and summing over ν. The left side of the result is identical with the left side of (3.8). By (3.7) the right side of the result is

$$\delta(x-y) + \int_{-(y+\delta)}^{y} \hat{K}(y,z)\delta(x-z)\, dz \, .$$

Thus for $x > y$ the right side of the result vanishes, and (3.8) follows.

We are now prepared to obtain an integral equation for $K(x,y)$, from which by (3.3) we can determine $V(x)$. Consider the relation (3.2) in the particular form

$$(3.10) \qquad v(k,x) = \alpha(k)\, e^{ikx} + \beta(k)\, e^{-ikx} + \int_{-(x+\delta)}^{x} K(x,z)\left[\alpha(k)\, e^{ikz} + \beta(k)\, e^{-ikz}\right] dz.$$

We multiply (3.10) by $\frac{e^{-iky}}{2\pi\, a(k)}$ and integrate over the contour C of (3.6). From (3.8) it is apparent that the left side of the resulting equation vanishes for $x > y$. We have then an equation of the form

$$0 = B(x+y) + K(x,y) + \int_{-(x+\delta)}^{x} K(x,z)\, B(y+z)\, dz\,, \qquad x > y \tag{3.11}$$

where

$$B(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)\, e^{-ik\tau} dk - i \sum_{\nu} b_{\nu}\, e^{-ik_{\nu}\tau}\,. \tag{3.12}$$

In (3.12) we have written b_ν for the residues of $b(k)$ at the poles k_ν of $b(k)$ in the upper half-plane, and we have used the fact that $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\tau}\, dk = \delta(\tau)$ as well as the definition $b(k) = \frac{\beta(k)}{\alpha(k)}$ to arrive at (3.11).

For every fixed value of x equation (3.11) is a Fredholm integral equation of the second kind for $K(x,y)$ when y ranges over the interval $-\infty < y < x$. The kernel $B(y+z)$ is given by (3.12) in terms of the reflection coefficient $b(k)$.

If (3.11) has a unique solution and we solve for $K(x,y)$, the solution must be the function $K(x,y)$ in (3.10). The necessary step remaining, therefore, is to prove the uniqueness of the solution of (3.11). We can do this by showing that the operator kernel $\delta(x-y) + B(x+y)$ is positive definite.

First we establish the identity

$$\delta(x-y) + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)\, e^{-ik(x+y)} dk = \frac{1}{2\pi}\int_{0}^{\infty} |a(k)|^2\, e^{ik(x-y)}\, dk + \frac{1}{2\pi}\int_{-\infty}^{0} \left[e^{ikx} + b(k)\, e^{-ikx}\right] \left[\overline{e^{iky} + b(k)\, e^{-iky}}\right] dk\,. \tag{3.13}$$

Here $a(k)$ is the transmission coefficient. Using (2.8) we can write (3.13) in the form

$$\delta(x-y) + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)\, e^{-ik(x+y)}dk = \frac{1}{2\pi}\int_{0}^{\infty} e^{ik(x-y)}dk - \frac{1}{2\pi}\int_{0}^{\infty} |b(k)|^2\, e^{ik(x-y)}dk$$
$$+ \frac{1}{2\pi}\int_{-\infty}^{0} e^{ik(x-y)}dk + \frac{1}{2\pi}\int_{-\infty}^{0} |b(k)|^2\, e^{-ik(x-y)}dk$$
$$+ \frac{1}{2\pi}\int_{-\infty}^{0} b(k)\, e^{-ik(x+y)} + \frac{1}{2\pi}\int_{-\infty}^{0} \overline{b(k)}\, e^{ik(x+y)}dk \,.$$

The sum of the first and third terms on the right side is $\delta(x-y)$, and the second and fourth terms on the right side cancel. We have finally

$$(3.14) \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)\, e^{-ik(x+y)}dk = \frac{1}{2\pi}\int_{-\infty}^{0} b(k)\, e^{-ik(x+y)}dk + \frac{1}{2\pi}\int_{-\infty}^{0} \overline{b(k)}\, e^{ik(x+y)}dk \,.$$

Changing variables in (2.4) and (2.7), from k to -k in the second terms on the right, we see that (2.14) is, indeed, an identity. Since our steps are obviously reversible, it follows that (3.13) is an identity.

The term $-i\sum_{\nu} b_{\nu}\, e^{-ik_{\nu}(x+y)}$ can be shown to be real and positive for all real x and y. The exponential factor is obviously real and positive, since the k_{ν} are pure imaginary quantities. Now $b_{\nu} = \dfrac{\beta(k_{\nu})}{\alpha'(k_{\nu})}$, by the definition of b_{ν} as a residue of a pole of b(k). From the known relations (see [9], Appendix II)

$$\alpha'(k_{\nu}) = -i\int_{-\infty}^{\infty} \hat{v}(k_{\nu},x)v(k_{\nu},x)\,dx, \quad v(k_{\nu},x) = \beta(k_{\nu})\hat{v}(k_{\nu},x)\,,$$

and from the fact that $\hat{v}(k_{\nu},x)$ is real for real x, it is seen without difficulty that $-i\, b_{\nu}$ is a real positive number.

Now if there were two solutions, $K_1(x,y)$ and $K_2(x,y)$, of (3.11), their difference $K_1 - K_2 = W(x,y)$ would be a solution of the corresponding homogeneous integral equation

$$\int_{-\infty}^{x} [\delta(y,z) + B(y+z)]\, W(x,z)\, dz = 0 . \tag{3.15}$$

According to (3.12) and (3.13) this can be written

$$\int_{-(x+\delta)}^{x} \left\{ \frac{1}{2\pi} \int_{0}^{\infty} |a(k)|^2 e^{ik(y-z)}\, dk + \frac{1}{2\pi} \int_{-\infty}^{0} \left[e^{iky} + b(k)\, e^{-iky}\right] \right.$$
$$\left. \times \left[\overline{e^{ikz} + b(k)\, e^{-ikz}}\right] dk - i \sum_{\nu} b_{\nu}\, e^{-ik_{\nu}(y+z)} \right\} W(x,z)\, dz = 0.$$

On multiplying through by $\overline{W(x,y)}$ and integrating with respect to y from $-(x+\delta)$ to x, and then interchanging integral signs, we obtain

$$\frac{1}{2\pi} \int_{0}^{\infty} |a(k)|^2 \left| \int_{-(x+\delta)}^{x} W(x,y) e^{iky}\, dy \right|^2 dk + \frac{1}{2\pi} \int_{-\infty}^{0} \left| \int_{-(x+\delta)}^{x} W(x,y) \left[e^{iky} + b(k) e^{-iky}\right] dy \right|^2 dk$$

$$+ \sum_{\nu} (-ib_{\nu}) \left| \int_{-(x+\delta)}^{x} W(x,y)\, e^{ik_{\nu}y}\, dy \right|^2 = 0 . \tag{3.16}$$

Since each term on the left side of (3.16) is positive, each term must be separately zero. Thus, for example

$$\int_{-(x+\delta)}^{x} W(x,y)\, e^{\pm iky}\, dy = 0,$$

whenever $|a(k)| \neq 0$. It follows that $W(x,y) = 0$ almost everywhere when $-(x+\delta) \leq y \leq x$.

4. Sufficient conditions for b(k) to be a reflection coefficient (the synthesis problem)

We have seen how to obtain an integral equation for the function K(x,y), from which we can determine V(x), if the reflection coefficient associated with

$V(x)$ is given. In the problem of synthesis, however, we are not only required to construct $V(x)$, but generally we must first fix upon an appropriate function $b(k)$ for the reflection coefficient, one that satisfies certain physical requirements. The question arises: if a function $b(k)$ is given, then by the process described in Section 3 can we always arrive at a function $V(x)$ corresponding to which $b(k)$ is the reflection coefficient? Undoubtedly, certain conditions must be imposed on $b(k)$ before this will be possible. We shall require, accordingly:

I. $b(k) = \overline{b(-k)}$ for real k;

II. $b(k)$ is analytic in k with no singularities in the upper half-plane except possibly a finite number of simple poles on the positive imaginary axis, and no singularities on the real axis except, perhaps, at zero;

III. The residues of $b(k)$ at its poles on the positive imaginary axis are pure imaginary numbers with positive imaginary parts;

IV. $|b(k)| \leq 1$ for all real k;

V. $b(k) = e^{-ik\delta}\, o(\frac{1}{k})$ in the upper half-plane;

VI. the Fourier transform $B(\tau)$ of $b(k)$ exists and is continuous in $-\infty < \tau < \infty$;

VII. $B(\tau)$ has piecewise continuous first and second derivatives in any finite interval.

These conditions are sufficient for $b(k)$ to be the reflection coefficient corresponding to some quantity $V(x)$ in (2.1), as we shall see. Condition V guarantees that $B(\tau) = 0$ for $\tau < -\delta$, and therefore, by (3.11), that $K(x,y) = 0$ for $x + y < -\delta$. This implies that $V(x) = 2\,\frac{dK(x,x)}{dx} = 0$ [see (3.3)] for $x < -\frac{\delta}{2}$. Conditions VI and VII along with (3.11) imply that $K(x,y)$ is continuous and has piecewise continuous first and second partial derivatives.

We shall now prove that $b(k)$ satisfying conditions I to VII is actually the reflection coefficient corresponding to some differential equation of the form (2.1) with a $V(x)$ obtained from $b(k)$ by solving (3.11) for $K(x,y)$ and setting $V(x) = 2\,\frac{dK(x,x)}{dx}$. The proof is divided into two parts. In the first part we show that $K(x,y)$ satisfies (3.3). We start with (3.11), which, because of V [observe that, since $x > y$, $-(y+\delta) > -(x+\delta)$], takes the form

$$0 = B(x+y) + K(x,y) + \int_{-(y+\delta)}^{x} K(x,z)\, B(y+z)\, dz; \qquad x > y,\ x > -(y+\delta). \tag{4.1}$$

Differentiating (4.1) twice with respect to x gives

$$\begin{aligned} 0 = B''(x+y) &+ K_{xx}(x,y) + B'(x+y)K(x,x) + B(x+y)\frac{d}{dx}K(x,x) \\ &+ B(x+y)\,K_x(x,x) + \int_{-(y+\delta)}^{x} K_{xx}(x,z)\, B(y+z)\, dz\,. \end{aligned} \tag{4.2}$$

Differentiating (4.1) twice with respect to y and integrating twice by parts gives, when we use the fact that $B(\tau)$ is continuous,

$$\begin{aligned} 0 = B''(x+y) &+ K_{yy}(x,y) + B'(x+y)\,K(x,x) - B(x+y)\,K_y(x,x) \\ &+ \int_{-(y+\delta)}^{x} K_{yy}(x,z)\, B(y+z)\, dz. \end{aligned} \tag{4.3}$$

Now we subtract (4.3) from (4.2) and add to this result the product of $-2\,\frac{dK(x,x)}{dx}$ with (4.1). We then have

$$0 = \left\{K_{xx}(x,y) - K_{yy}(x,y) - 2\,\frac{dK(x,x)}{dx}\,K(x,y)\right\}$$

(4.4)
$$+\int_{-(y+\delta)}^{x} \left\{K_{xx}(x,z) - K_{yy}(x,z) - 2\,\frac{dK(x,x)}{dx}\,K(x,z)\right\} B(y+z)\,dz.$$

It follows from the uniqueness theorem already proved that

(4.5)
$$K_{xx}(x,y) - K_{yy}(x,y) - 2\,\frac{dK(x,x)}{dx}\,K(x,y) = 0;$$

i.e., $K(x,y)$ satisfies (3.3). Further, a solution $u(k,x)$ of (2.1) is given by (3.2) when $u_0(k,x)$ satisfies (3.1). The function $u(k,x)$ given by (3.2) is evidently continuous, and because $K(x,y)$ is continuous and partial derivatives of $K(x,y)$ are piecewise continuous, $u(k,x)$ has a continuous derivative.

For the second part of the proof we must show that the solution of (2.1) given by (3.2) when $u_0(k,x) = e^{ikx} + b(x)\,e^{-ikx}$ is the one which has the behavior $u(k,x) \sim a(k)\,e^{ikx}$ for $x \to +\infty$. It is difficult to ascertain directly the behavior as $x \to +\infty$ of a solution $u(k,x)$ of (2.1) when $u(k,x)$ is given in the form (3.2), but its behavior as $x \to -\infty$ is just $u_0(k,x)$. We shall therefore investigate the character of our solution $u(k,x)$, which approaches $e^{ikx}+b(k)\,e^{-ikx}$ as $x \to -\infty$, by comparing it with that (unique) solution $\hat{u}(k,x)$ of (2.1) which has the behavior e^{-ikx} as $x \to -\infty$. We know: if and only if

(4.6)
$$\frac{1}{2\pi}\int_C u(k,x)\,\hat{u}(k,y)\,dk = \delta(x-y)\ ,$$

$u(k,x)$ has the behavior

$$
(4.7) \qquad u(k,x) \sim \begin{cases} e^{ikx} + b(k)\, e^{-ikx}, & x \to -\infty \\ a(k)\, e^{ikx}, & x \to +\infty, \end{cases}
$$

for the general spectral theory of (2.1) shows that (4.6) is a necessary condition and it follows immediately on multiplying (4.6) by the conjugate function [the one which has the behavior (4.7) of $\hat{u}(k,y)$] and integrating with respect to y that (4.6) is also a sufficient condition. We write then

$$
(4.8) \qquad \begin{aligned} u(k,x) &= e^{ikx} + b(k)\, e^{-ikx} + \int_{-(x+\delta)}^{x} K(x,z) \left[e^{ikz} + b(k)\, e^{-ikz}\right] dz \\ \hat{u}(k,y) &= e^{-iky} + \int_{-(x+\delta)}^{x} K(x,z)\, e^{-ikz}\, dz \quad . \end{aligned}
$$

If we take the product of these two expressions and perform the integration indicated in (4.6), making use of (3.11), we find that (4.6) is satisfied. Thus the solution $u(k,x)$ of (2.1) has the required asymptotic behavior and the quantity $b(k)$ is, indeed, the reflection coefficient corresponding to $V(x) = 2\,\frac{dK(x,x)}{dx}$.

5. Example

Let $b(k) = -\frac{1}{k^2+1}$ be the reflection coefficient for some as yet unknown function $V(x)$ in (2.1). The sufficient conditions I-VII on $b(k)$ (see Sec. 4)) are easily verified for this case. Also the function $B(\tau)$ is easily calculated by a straightforward application of the theory of residues:

$$
(5.1) \qquad B(\tau) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ik}}{k^2+1}\, dk + \frac{1}{2}\, e\tau = \begin{cases} \sinh \tau, & \tau \geq 0 \\ 0, & \tau \leq 0 \ . \end{cases}
$$

Then equation (3.11) becomes in this case

$$(5.2)\qquad 0 = \sinh\,(x+y) + K(x,y) + \int_{-y}^{x} K(x,z)\sinh(y+z)\,dz, \qquad x > |y| .$$

We can obtain a differential equation for $K(x,y)$ by differentiating (5.2) twice with respect to y. A single derivative gives

$$(5.3)\qquad 0 = \cosh\,(x+y) + K_y(x,y) + \int_{-y}^{x} K(x,z)\cosh(y+z)\,dz,$$

and a second derivative gives

$$(5.4)\qquad 0 = \sinh\,(x+y) + K_{yy}(x,y) + K(x,-y) + \int_{-y}^{x} K(x,z)\sinh(y+z)\,dz.$$

The integral term of (5.4) is the same as that of (5.2). We can therefore eliminate this term, obtaining

$$(5.5)\qquad K_{yy}(x,y) + K(x,-y) - K(x,y) = 0.$$

It is evident from (5.5) that $K_{yy}(x,y)$ is an odd function of y. Thus

$$(5.6)\qquad K(x,y) = - K(x,-y) + a(x)y + 2b(x) ,$$

where $a(x)$ and $b(x)$ are arbitrary functions of x alone. Then (5.5) becomes, after we substitute for $K(x,-y)$ from (5.6),

$$(5.7)\qquad K_{yy}(x,y) = 2K(x,y) - a(x)y - b(x).$$

The general solution of (5.7) is

$$(5.8)\qquad K(x,y) = \frac{a(x)y + 2b(x)}{2} + \frac{A(x)}{2}\, e^{\sqrt{2}\,y} + \frac{B(x)}{2}\, e^{-\sqrt{2}\,y} ,$$

where $A(x)$ and $B(x)$ are new arbitrary functions of x alone. After substituting the value of $K(x,y)$ obtained from (5.8) into (5.5), we see that $a(x) = 0$ and $A(x) = B(x)$. Thus, by (5.8)

$$K(x,y) = b(x) + A(x) \sinh \sqrt{2}\, y \quad . \tag{5.9}$$

To determine the functions $b(x)$ and $A(x)$ we substitute (5.9) into (5.2). Then we have the result

$$A(x) = - \frac{\operatorname{sech} \sqrt{2}\, x}{\sqrt{2}} \, , \qquad b(x) = - \frac{\tanh \sqrt{2}\, x}{\sqrt{2}} \quad . \tag{5.10}$$

From (5.9) we now have

$$K(x,y) = \begin{cases} - \dfrac{1}{\sqrt{2}} \; \dfrac{\sinh \sqrt{2}\, x + \sinh \sqrt{2}\, y}{\cosh \sqrt{2}\, y} \, , & y \geq -x \\ 0 \, , & y \leq -x \; . \end{cases} \tag{5.11}$$

By (3.3) we finally have

$$V(x) = \begin{cases} -4 \operatorname{sech}^2 \sqrt{2}\; x \, , & x > 0 \\ 0 \, , & x < 0 \; . \end{cases} \tag{5.12}$$

For this $V(x)$, the solution $u(k,x)$ of (2.1) having the asymptotic behavior given by (2.6) and thus exhibiting the reflection coefficient $b(k) = - \dfrac{1}{k^2 + 1}$ can be written immediately through the use of (3.2) and (5.11):

$$\begin{aligned} u(k,x) = e^{ikx} - \frac{1}{k^2+1}\; e^{-ikx} - \frac{\operatorname{sech} \sqrt{2}\, x}{\sqrt{2}} \\ \cdot \int_{-x}^{x} \left[\sinh \sqrt{2}\, x + \sinh \sqrt{2}\, y \right] \left[e^{iky} - \frac{1}{k^2+1}\, e^{-iky} \right] dy \; . \end{aligned} \tag{5.13}$$

After performing the indicated integration and simplifying the right side of (5.13) we obtain

$$(5.14) \quad u(k,x) = \begin{cases} \left[\dfrac{k^2 + \sqrt{2}\, ik \tanh \sqrt{2}\, x}{k^2+1}\right] e^{ikx}, & x \geq 0 \\[2ex] e^{ikx} - \dfrac{1}{k^2+1}\, e^{-ikx}, & x \leq 0 . \end{cases}$$

Since, as $x \to +\infty$, $u(k,x) \sim \dfrac{k^2 + \sqrt{2}\, ik}{k^2+1}\, e^{ikx}$, the transmission coefficient is given by

$$(5.15) \qquad a(k) = \frac{k^2 + \sqrt{2}\, ik}{k^2+1} \quad .$$

Obviously $a(k)$ and $b(k)$ satisfy the required conditions

$$a(-k) = \overline{a(k)},\ b(-k) = \overline{b(k)}, \quad |a(k)|^2 + |b(k)|^2 = 1 \text{ for real } k.$$

We might note in passing that if $V(x) = -4 \operatorname{sech}^2 \sqrt{2}\, x$ for the entire interval $-\infty < x < \infty$, it is 'transparent'; that is, there exists a solution of (2.1) (we can obtain this solution from (5.14)) which has the behavior $u \sim e^{ikx}$ as $x \to -\infty$ and the behavior $u \sim \dfrac{k + \sqrt{2}\, i}{k - \sqrt{2}\, i}\, e^{ikx}$ as $x \to +\infty$, so that no reflected wave exists for an incident wave from the left. The amplitude of the incident wave remains one as the wave approaches $+\infty$, but the phase changes by an amount depending on the frequency.

6. Solution of a general class of problems

The method employed in Section 5 of solving integral equation (5.2) suggests that a whole class of problems associated with reflection coefficients of a certain form might be solved explicitly.

Suppose we have a reflection coefficient $b(k)$ satisfying conditions I to VII of Section 4 and suppose also that $b(k)$ is a meromorphic function with a finite number of poles in either half-plane. The function $B(\tau)$ then

has the form

$$(6.1) \qquad B(\tau) = \begin{cases} \sum_{n=1}^{N} b_n \, e^{-ik_n \tau} , & \tau \geq -\delta \\ 0 , & \tau \leq -\delta \end{cases} .$$

Accordingly the integral equation (3.11) for $K(x,y)$ becomes

$$(6.2) \qquad 0 = \sum_{n=1}^{N} b_n \, e^{-ik_n(x+y)} + K(x,y) + \sum_{n=1}^{N} b_n \int_{-(y+\delta)}^{x} K(x,z) \, e^{-ik_n(y+z)} dz .$$

We can obtain a new equation from (6.2) by differentiating with respect to y; each time we differentiate we obtain another equation. Let us differentiate N times, forming N + 1 equations altogether, including (6.2) itself. The new equations have the form

$$(6.3) \qquad \begin{aligned} 0 = \sum_{n=1}^{N} (-ik_n)^{\nu} b_n e^{-ik_n(x+y)} + K^{(\nu)}(x,y) + \sum_{n=1}^{N} (-ik_n)^{\nu} b_n e^{-ik_n y} \int_{-(y+\delta)}^{x} K(x,z) e^{-ik_n z} dz \\ + \sum_{n=1}^{N} b_n \, e^{ik_n \delta} \sum_{\mu=1}^{\nu-1} (-ik_n)^{\mu} \, K^{(\nu-\mu-1)}(x,-(y+\delta)); \qquad \nu = 1, \ldots, N , \end{aligned}$$

where $K^{(\nu)}(x,y)$ is written for the ν^{th} y-derivative of $K(x,y)$ and where we have used the fact in (6.1) that $B(-\delta) = \sum_{n=1}^{N} b_n e^{ik_n \delta} = 0$. We can eliminate the N quantities

$$\int_{-(y+\delta)}^{x} K(x,z) \, e^{-ik_n z} \, dz$$

from equations (6.3) and (6.2) obtaining a single N^{th}-order differential equation in $K(x,y)$ and $K(x,-(y+\delta))$ of the form:

$$(6.4) \qquad 0 = P(x,y) + \sum_{n=0}^{N} Q_n(y) \, K^{(n)}(x,y) + \sum_{n=0}^{N-2} R_n(y) \, K^{(n)} \, (x,-(y+\delta)) .$$

By substituting $-y-\delta$ for y in (6.4) we obtain another differential equation. By differentiating (6.4) and then performing this substitution again we obtain a new equation. If we differentiate $N - 2$ times and perform the substition each time, we arrive at a system of equations involving $K(x,y)$ and its derivatives and $K(x,-(y+\delta))$ and its derivatives. It is now possible to eliminate $K(x,-(y+\delta))$ and its derivatives and finally obtain a single differential equation of order $2N - 2$ for $K(x,y)$. In the example in Section 5, $N=2$, and consequently we obtain a second-order differential equation for $K(x,y)$ in this case.

In most cases we can solve the integral equation (3.11) by a more direct method, however. Let us assume that $K(x,y)$ has the form:

$$K(x,y) = \sum_{\nu=-N}^{N} f_\nu(x)\, e^{\alpha_\nu y} , \tag{6.5}$$

where the sum extends over all integers between $-N$ and N <u>except zero</u>, and where α_ν and $f_\nu(x)$ are quantities to be determined. If we substitute (6.5) into (3.11) we have

$$0 = \sum_{n=1}^{N} b_n\, e^{-ik_n(x+y)} + \sum_{\nu=-N}^{N} f_\nu(x) e^{\alpha_\nu y} + \sum_{n=1}^{N} \sum_{\nu=-N}^{N} b_n e^{-ik_n y} f_\nu(x) \int_{-(y+\delta)}^{x} e^{(\alpha_\nu - ik_n)z}\, dz$$

$$= \sum_{n=1}^{N} b_n\, e^{-ik_n(x+y)} + \sum_{\nu=-N}^{N} f_\nu(x) e^{\alpha_\nu y} + \sum_{n=1}^{N} \sum_{\nu=-N}^{N} b_n e^{-ik_n y} f_\nu(x) \left[\frac{e^{(\alpha_\nu - ik_n)x} - e^{-(y+\delta)(\alpha_\nu - ik_n)}}{\alpha_\nu - ik_n}\right]$$

$$= \sum_{n=1}^{N} b_n e^{-ik_n(x+y)} + \sum_{n=1}^{N} b_n\, e^{-ik_n(x+y)} \sum_{\nu=-N}^{N} \frac{e^{\alpha_\nu x} f_\nu(x)}{\alpha_\nu - ik_n} + \sum_{\nu=-N}^{N} f_\nu(x)\, e^{\alpha_\nu y}$$

$$- \sum_{\nu=-N}^{N} e^{-\alpha_\nu y - \alpha_\nu \delta} f_\nu(x) \sum_{n=1}^{N} \frac{b_n\, e^{ik_n \delta}}{\alpha_\nu - ik_n} .$$

This form suggests that we assume $\alpha_\nu = -\alpha_{-\nu}$; then

$$(6.6) \qquad 0 = 1 + \sum_{\nu=-N}^{N} \frac{f_\nu(x)\, e^{\alpha_\nu x}}{\alpha_\nu - ik_n} ; \qquad n = 1, \ldots, N,$$

and

$$(6.7) \qquad 0 = f_\nu(x) + f_{-\nu}(x)\, e^{\alpha_\nu \delta} \sum_{n=1}^{N} \frac{b_n e^{ik_n \delta}}{\alpha_\nu + ik_n} ; \qquad \nu = 1, \ldots, N .$$

A similar set of equations goes with (6.7) and can be deduced from it by substituting $-\nu$ for ν. In order to get a non-trivial solution $f_\nu(x)$ of the resulting homogeneous system, the coefficient determinant must be set equal to zero:

$$(6.8) \qquad \left[\sum_{n=1}^{N} \frac{b_n e^{ik_n \delta}}{ik_n - \alpha_\nu} \right] \left[\sum_{n=1}^{N} \frac{b_n e^{ik_n \delta}}{ik_n + \alpha_\nu} \right] - 1 = 0 .$$

Equation (6.8) reduces to an N^{th}-degree algebraic equation for α_ν; thus in general there are N values which α_ν can have. It is evident from (6.8) that if α_ν is a solution then $\alpha_{-\nu} = -\alpha_\nu$ is also a solution. When α_ν is known, equations (6.7) can be used to give $f_{-\nu}(x)$ in terms of $f_\nu(x)$. After substituting this result in (6.6) we have

$$(6.9) \qquad -1 = \sum_{\nu=1}^{N} \left[\frac{e^{\alpha_\nu x}}{\alpha_\nu - ik_n} + \frac{e^{-\alpha_\nu (x+\delta)}}{\alpha_\nu + ik_n} \sum_{\mu=1}^{N} \frac{b_n e^{ik_n \delta}}{ik_n - \alpha_\nu} \right] f_\nu(x); \quad n = 1, \ldots, N.$$

Equations (6.9) can be solved for $f_\nu(x)$ provided that the coefficient determinant does not vanish. The coefficient determinant will surely vanish if any two values of α_ν are the same as ν goes from one to N. Thus we need the condition generally that (6.8) have no multiple roots, except perhaps $\alpha_\nu = 0$, which can be a double root.

If it turns out that equations (6.9) have no solutions, we must assume a different form for $K(x,y)$:

$$(6.10) \qquad K(x,y) = \sum_{\nu=-N}^{N} f_\nu(x)\, P_\nu(y)\, e^{\alpha_\nu y} ,$$

where the $P_\nu(y)$ are polynomials in y. A similar procedure to the one just described should lead to a determination of the α_ν, $f_\nu(x)$ and the $P_\nu(y)$ and thus of $K(x,y)$.

7. The second differential equation

Often the differential equation of interest in scattering problems is not (2.1) but rather an equation of the form

$$(7.1) \qquad \frac{d^2u(k,x)}{dx^2} + k^2n^2(x)\, u(k,x) = 0 ,$$

where $n(x)$ is real, continuous and has a continuous derivative in $-\infty < x < \infty$. The solution of (7.1) can also be interpreted as a wave if $n(x)$ approaches a constant as $x \to \pm\infty$, and a reflection coefficient $b(k)$ can again be defined. The problem of determining $n^2(x)$ from the reflection coefficient can be dealt with by a consideration of an equation of the form (2.1) corresponding in a certain way to (7.1).

Suppose that we have a differential equation of the form (2.1):

$$(7.2) \qquad w''(z) + [k^2 - V(z)]\, w(z) = 0 ,$$

where the primes indicate differentiation with respect to z. Let us make the substitution $x = x(z)$ in (7.2). We have

$$(7.3) \qquad \ddot{w} + \frac{x''}{x'^2}\, \dot{w} + \frac{(k^2 - V)}{x'^2}\, w = 0,$$

where again the primes indicate differentiation with respect to z and the dots with respect to x. Now let us make the substitution $w = \frac{u(k,x)}{\sqrt{x'}}$ in (7.3):

$$(7.4) \qquad \ddot{u} + \left[\frac{k^2}{x'^2} + \frac{3x''^2}{4x'^4} - \frac{x'''}{2x'^3} - \frac{V}{x'^2} \right] u = 0 .$$

If the second term in the bracketed factor of u in (7.4) is zero, (7.4) will have the form (7.1) with $n(x) = \frac{1}{x'}$. Moreover, if x(z) is a monotonic function of z and independent of k, the reflection coefficient is the same for equations (7.2) and (7.4). Therefore, we can find $n(x) = \frac{1}{x'}$ if we can find x(z). We assume that V(z) is constructed by the method of the previous sections, and now it can be used to obtain x(z).

We set the second term in the brackets in (7.4) equal to zero, and this implies

$$3x''^2 - 2x' x''' - 4x'^2 V(z) = 0$$

or

$$-4 \frac{x'}{x''} V - 2 \frac{x'''}{x''} + 3 \frac{x''}{x'} = 0$$

or

$$(7.5) \qquad 2 \frac{d}{dz} (\log x'') - 3 \frac{d}{dz} (\log x') = \frac{-4V}{\frac{d}{dz} (\log x')} .$$

Now we define $\omega = \frac{x''}{x'} = \frac{d}{dz} (\log x')$, from which we have

$$(7.6) \qquad \frac{d}{dz} (\log x'') = \omega + \frac{\omega'}{\omega} .$$

Substituting in (7.5) gives us

$$(7.7) \qquad \omega' - \frac{1}{2} \omega^2 = -2V .$$

Equation (7.7) is a Ricatti equation and can be written as a linear second-order differential equation by means of the transformations

$$\omega = -2W; \quad W = \frac{U'}{U} .$$

We have finally

$$U''(z) - V(z)\,U(z) = 0, \tag{7.8}$$

where $x' = \dfrac{C}{U^2(z)}$ and C is an arbitrary contant; thus

$$x = C \int^{z} \frac{dz}{U^2(z)} \; . \tag{7.9}$$

After we have found V(z) by the methods of the previous sections, we can use (3.2) with k set equal to zero to obtain a solution U(z) of (7.8). Then by means of (7.8) and (7.9) we get

$$n(x) = \frac{1}{x'\big(z(x)\big)} = U^2\big(z(x)\big) \; . \tag{7.10}$$

The case where n(x) = constant in (7.1) corresponds to the case where V(z) = 0 in (7.2). According to (7.9), x would then be proportional to z or to $\frac{1}{z+A}$, where A is an arbitrary constant. The second case, however, leads in general to a singular n(z).

In each problem it must be decided which of the solutions of (7.8) to use, according to the physical situation. Obviously, n(x) is not determined uniquely by the reflection coefficient as is V(z).

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